# An Approximate Frequency-Integrated Transfer Equation

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We look for a frequency-integrated transfer equation such that it reduces to a desired set of frequency-integrated moments. This is the form of the frequency-integrated transfer equation used by the Athena++ code. See also *Foundations of Radiation Hydrodynamics* by Mihalas & Mihalas section 82 for related ideas.

## 0.1 Definitions

The specific energy density of radiation $E_\nu$, the radiative energy flux $\vec{F}_\nu$, and radiation pressure tensor $\mathbf{P}_\nu$, whose components are given by

$$J_\nu = \frac{1}{4\pi}\int I_\nu(\hat{n})d\Omega \qquad E_\nu = \frac{1}{c}\int I_\nu(\hat{n})d\Omega = \frac{4\pi J_\nu}{c} \tag{1}$$

$$H_\nu^i = \frac{1}{4\pi}\int \mu_i I_\nu(\hat{n})d\Omega \qquad F_\nu^i = \int \mu_i I_\nu(\hat{n})d\Omega = 4\pi H_\nu^i \tag{2}$$

$$K_\nu^{ij} = \frac{1}{4\pi}\int \mu_i\mu_j I_\nu(\hat{n})d\Omega \qquad P_\nu^{ij} = \frac{1}{c}\int \mu_i\mu_j I_\nu(\hat{n})d\Omega = \frac{4\pi K_\nu^{ij}}{c} \tag{3}$$

where $\mu_i = \hat{n}\cdot\hat{x}_i$, with $\hat{x}_i$ some basis vector. Frequency-integrated quantities will be denoted by dropping the $\nu$ subscript, $B \equiv \int B_\nu d\nu = \sigma T^4/\pi$ for example. Furthermore define Planck opacity

$$\kappa_p \equiv \frac{1}{B}\int \kappa_{a,\nu} B_\nu d\nu \tag{4}$$

and Rosseland mean opacities

$$\kappa_s \equiv \left(\frac{\int \partial B/\partial T d\nu}{\int \frac{\partial B/\partial T}{\kappa_{s,\nu}} d\nu}\right)^{-1} \tag{5}$$

$$\kappa_a \equiv \left(\frac{\int \partial B/\partial T d\nu}{\int \frac{\partial B/\partial T}{\kappa_{a,\nu}} d\nu}\right)^{-1} \tag{6}$$

## 0.2 Moments of the transfer equation

The time-dependent transfer equation is

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{n}\cdot\nabla I_\nu = (\sigma_{a,\nu} + \sigma_{s,\nu})\,(S_\nu - I_\nu) \tag{7}$$

We can plug in the source function $S_\nu$ for isotropic scattering,

$$S_\nu = \frac{\sigma_{a,\nu} B_\nu}{\sigma_{a,\nu} + \sigma_{s,\nu}} + \frac{\sigma_{s,\nu} J_\nu}{\sigma_{a,\nu} + \sigma_{s,\nu}} \tag{8}$$

to get

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{n}\cdot\nabla I_\nu = \sigma_{a,\nu} B_\nu + \sigma_{s,\nu} J_\nu - (\sigma_{a,\nu} + \sigma_{s,\nu})\, I_\nu \tag{9}$$

### 0.2.1 Zeroth moment transfer equation

We can take the zeroth moment of this equation,

$$\underbrace{\int \frac{1}{c}\frac{\partial I_\nu}{\partial t} d\Omega}_{\frac{4\pi}{c}\frac{\partial J_\nu}{\partial t}} + \underbrace{\int \hat{n}\cdot\nabla I_\nu d\Omega}_{\nabla\cdot\vec{F}_\nu} = \underbrace{\int \sigma_{a,\nu} B_\nu d\Omega}_{4\pi\sigma_{a,\nu} B_\nu} + \underbrace{\int \sigma_{s,\nu} J_\nu d\Omega}_{4\pi\sigma_{s,\nu} J_\nu} - \underbrace{\int (\sigma_{a,\nu} + \sigma_{s,\nu})\, I_\nu d\Omega}_{4\pi(\sigma_{a,\nu}+\sigma_{s,\nu})J_\nu} \tag{10}$$

to get the frequency-dependent zeroth moment transfer equation

$$\frac{\partial E_\nu}{\partial t} + \nabla\cdot\vec{F}_\nu = 4\pi\sigma_{a,\nu}(B_\nu - J_\nu) \tag{11}$$

We can try to get a frequency-integrated form by integrating each term over frequency:

$$\int \frac{\partial E_\nu}{\partial t} d\nu + \int \nabla\cdot\vec{F}_\nu d\nu = \int 4\pi\sigma_{a,\nu}(B_\nu - J_\nu) d\nu \tag{12}$$

because $B_\nu$ is a known function, we can take care of the $B_\nu$ integral by leveraging the Planck opacity, but the $J_\nu$ term can't be integrated over without solving the full frequency-dependent transfer.

$$\frac{\partial E}{\partial t} + \nabla\cdot\vec{F} = 4\pi\sigma_p B - 4\pi \int \sigma_{a,\nu} J_\nu d\nu \tag{13}$$

But maybe we can approximate $\sigma_{a,\nu}$ with a frequency-integrated opacity that isn't so bad. To figure out what frequency-integrated opacity we should use we look at the frequency-dependent zeroth moment transfer equation (11) and its limiting behavior. At high optical depth $I_\nu \to B_\nu$ and the source term $4\pi\sigma_{a,\nu}(B_\nu - J_\nu) \to 0$ and the precise value of $\sigma_{a,\nu}$ is not so important. At low optical depth however, $B_\nu$ and $J_\nu$ can be very different. To maximize correctness in all regimes then we ought to use a frequency-integrated opacity which is more concerned with getting the right absorption in the optically thin regime i.e. the Planck opacity $\sigma_p \equiv \rho\kappa_p$. With this in mind, our approximately correct frequency-integrated zeroth moment equation is:

$$\boxed{\frac{\partial E}{\partial t} + \nabla\cdot\vec{F} = 4\pi\sigma_p(B - J)} \tag{14}$$

### 0.2.2 First moment transfer equation

We can also take the first moment of the transfer equation

$$\begin{aligned}\underbrace{\int \frac{\mu_i}{c}\frac{\partial I_\nu}{\partial t}d\Omega}_{\frac{4\pi}{c}\frac{\partial H^i_\nu}{\partial t}} + \underbrace{\int \mu_i \hat{n}\cdot\nabla I_\nu d\Omega}_{4\pi\nabla_j\cdot\mathbf{K^{ij}}_\nu} \\ = \underbrace{\int \mu_i\sigma_{a,\nu}B_\nu d\Omega}_{0} + \underbrace{\int \mu_i\sigma_{s,\nu}J_\nu d\Omega}_{0} - \underbrace{\int \mu_i\left(\sigma_{a,\nu}+\sigma_{s,\nu}\right)I_\nu d\Omega}_{4\pi(\sigma_{a,\nu}+\sigma_{s,\nu})H^i_\nu}\end{aligned} \tag{15}$$

to get a vector equation for the frequency-dependent first order transfer equation,

$$\frac{1}{c}\frac{\partial \vec{F}_\nu}{\partial t} + c\nabla\cdot\mathbf{P}_\nu = -(\sigma_{a,\nu}+\sigma_{s,\nu})\vec{F}_\nu \tag{16}$$

We again attempt to frequency integrate this

$$\int \frac{1}{c}\frac{\partial \vec{F}_\nu}{\partial t}d\nu + \int c\nabla\cdot\mathbf{P}_\nu d\nu = -\int(\sigma_{a,\nu}+\sigma_{s,\nu})\vec{F}_\nu d\nu \tag{17}$$

but are stuck with the source term because we don't a priori know the form of $F_\nu$

$$\frac{1}{c}\frac{\partial \vec{F}}{\partial t} + c\nabla\cdot\mathbf{P} = -\int(\sigma_{a,\nu}+\sigma_{s,\nu})\vec{F}_\nu d\nu \tag{18}$$

We opt for opacities that recover the appropriate fluxes in the diffusion limit by taking Rosseland mean opacities $\sigma_a \equiv \rho\kappa_a$ and $\sigma_s \equiv \rho\kappa_s$. (Note: it is not entirely precise to use the split scattering and absorption Rosseland means when typically the the Rosseland mean includes both scattering and absorption, i.e. $1/\sigma_a + 1/\sigma_s \neq 1/(\sigma_a+\sigma_s)$) In any case we make the first-moment frequency integrated equation to look like

$$\boxed{\frac{1}{c}\frac{\partial \vec{F}}{\partial t} + c\nabla\cdot\mathbf{P} = -(\sigma_a+\sigma_s)\vec{F}} \tag{19}$$

## 0.3 Frequency-integrated form of transfer equation

We now look for a frequency integrated form of

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{n}\cdot\nabla I_\nu = \sigma_{a,\nu}B_\nu + \sigma_{s,\nu}J_\nu - \left(\sigma_{a,\nu}+\sigma_{s,\nu}\right)I_\nu \tag{20}$$

such that two boxed equations are recovered when taking moments. We can supply our frequency integrated transfer equation with arbitrary coefficients $(X, Y, Z)$ and then take moments to figure out the required coefficients.

$$\frac{1}{c}\frac{\partial I}{\partial t} + \hat{n}\cdot\nabla I = XB + YJ + ZI \tag{21}$$

**Frequency Integrated Zeroth Moment**:

$$\frac{\partial E}{\partial t} + \nabla \cdot \vec{F} = XB + (Y + Z)J \tag{22}$$

**Frequency Integrated First Moment**:

$$\frac{1}{c}\frac{\partial \vec{F}}{\partial t} + c\nabla \cdot \mathbf{P} = Z\vec{F} \tag{23}$$

which upon comparing with the boxed moment equations yields the conditions:

$$\begin{aligned} X &= \sigma_p \\ Y + Z &= -\sigma_p \\ Z &= -\left(\sigma_a + \sigma_s\right) \end{aligned}$$

so that the frequency integrated transfer equation ought to be:

$$\boxed{\frac{1}{c}\frac{\partial I}{\partial t} + \hat{n} \cdot \nabla I = \sigma_p B + (\sigma_a + \sigma_s - \sigma_p)J_\nu - \left(\sigma_a + \sigma_s\right) I_\nu} \tag{24}$$

if we would like it to recover the frequency-integrated zeroth and first moments given above upon taking appropriate integrals.